

# Nonlinear Spinor Fields in Bianchi type-III spacetime

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Within the scope of Bianchi type-III spacetime we study the role of spinor field on the evolution of the Universe as well as the influence of gravity on the spinor field. In doing so we have considered a polynomial type of nonlinearity. In this case the spacetime remains locally rotationally symmetric and anisotropic all the time. It is found that depending on the sign of nonlinearity the models allows both accelerated and oscillatory modes of expansion. The non-diagonal components of energy-momentum tensor though impose some restrictions on metric functions and components of spinor field, unlike Bianchi type I, V and  $VI_0$  cases, they do not lead to vanishing mass and nonlinear terms of the spinor field.

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## I. INTRODUCTION

Discovery and further reconfirmation of the existence of the late time accelerated mode of expansion [1, 2] have given rise to a number of alternative studies of the evolution of the Universe.

Though the models with  $\Lambda$ -term [3–5], quintessence [6–11], Chaplygin gas [12–19] etc. retain their position as the prime candidates to explain this phenomenon, nevertheless some other models of dark energy are also proposed. After some remarkable works by different authors [20–34], showing the important role of spinor field in the evolution of the Universe, it has been extensively used to model the dark energy. This success is directly related to its ability to answer some fundamental questions of modern cosmology: (i) Problem of initial singularity and its possible elimination [22–24, 26, 27, 35–38]; (ii) problem of isotropization [24, 26, 29, 36, 39] and (iii) late time acceleration of the Universe [28, 30–32, 35–38, 40, 41]. Moreover recently it was found that the spinor field can also describe the different characteristics of matter from ekpyrotic matter to phantom matter, as well as Chaplygin gas [42–46].

In some recent studies [47–49] it was shown that due to its specific behavior in curve spacetime the spinor field can significantly change not only the geometry of spacetime but itself as well. The existence of nontrivial non-diagonal components of the energy-momentum tensor plays a vital role in this matter. In [47, 48] it was shown that depending on the type restriction imposed on the non-diagonal components of the energy-momentum tensor, the initially Bianchi type-I evolve into a LRS Bianchi type-I spacetime or FRW one from the very beginning, whereas the model may describe a general Bianchi type-I spacetime but in that case the spinor field becomes massless and linear. The same thing happens for a Bianchi type- $VI_0$  spacetime, i.e., the geometry of Bianchi type- $VI_0$  spacetime does not allow the existence of a massive and nonlinear spinor field in some particular cases [49]. In this paper we study the role of spinor field on the evolution of a Bianchi type-III spacetime.

The purpose of considering the anisotropic cosmological models lies on the fact that recent observational data from Cosmic Background Explorer's differential Radiometer have detected and measured cosmic microwave background anisotropies in different angular scale. We consider here Bianchi type III model due to some special interest of cosmologist towards it due to its peculiar geometric interpretation [50]. Yadav et al. [51], Pradhan et al. [52, 53] have recently studied homogeneous and anisotropic Bianchi type-III spacetime in context of massive strings. Recently Yadav [54] has obtained Bianchi type-III anisotropic DE models with constant deceleration parameter. In this paper, they have investigated a new anisotropic Bianchi type-III DE model with variable  $\omega$  without assuming constant deceleration parameter. A spinor description of dark energy within the scope of a BIII model was given in [55]. Further the accelerating dark energy models of the Universe within the scope of Bianchi type-III spacetime were studied in [56–58]. Bianchi type III cosmological models with varying  $\Lambda$  term was studied in [59].

## II. BASIC EQUATION

Let us consider the case when the anisotropic spacetime is filled with nonlinear spinor field. The corresponding action can be given by

$$\mathcal{S}(g; \psi, \bar{\psi}) = \int L \sqrt{-g} d\Omega \quad (2.1)$$

with

$$L = L_g + L_{sp}. \quad (2.2)$$

Here  $L_g$  corresponds to the gravitational field

$$L_g = \frac{R}{2\kappa}, \quad (2.3)$$

where  $R$  is the scalar curvature,  $\kappa = 8\pi G$ , with  $G$  being Newton's gravitational constant and  $L_{sp}$  is the spinor field Lagrangian.

### A. Gravitational field

The gravitational field in our case is given by a Bianchi type-III anisotropic spacetime:

$$ds^2 = dt^2 - a_1^2 e^{-2mx_3} dx_1^2 - a_2^2 dx_2^2 - a_3^2 dx_3^2, \quad (2.4)$$

with  $a_1$ ,  $a_2$  and  $a_3$  being the functions of time only and  $m$  is some arbitrary constant.

The nontrivial Christoffel symbols for (2.4) are

$$\begin{aligned} \Gamma_{01}^1 &= \frac{\dot{a}_1}{a_1}, \quad \Gamma_{02}^2 = \frac{\dot{a}_2}{a_2}, \quad \Gamma_{03}^3 = \frac{\dot{a}_3}{a_3}, \\ \Gamma_{11}^0 &= a_1 \dot{a}_1 e^{-2mx_3}, \quad \Gamma_{22}^0 = a_2 \dot{a}_2, \quad \Gamma_{33}^0 = a_3 \dot{a}_3, \\ \Gamma_{31}^1 &= -m, \quad \Gamma_{11}^3 = \frac{ma_1^2}{a_3^2} e^{-2mx_3}. \end{aligned} \quad (2.5)$$

The nonzero components of the Einstein tensor corresponding to the metric (2.4) are

$$G_1^1 = -\frac{\ddot{a}_2}{a_2} - \frac{\ddot{a}_3}{a_3} - \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3}, \quad (2.6a)$$

$$G_2^2 = -\frac{\ddot{a}_3}{a_3} - \frac{\ddot{a}_1}{a_1} - \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} + \frac{m^2}{a_3^2}, \quad (2.6b)$$

$$G_3^3 = -\frac{\ddot{a}_1}{a_1} - \frac{\ddot{a}_2}{a_2} - \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2}, \quad (2.6c)$$

$$G_0^0 = -\frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} - \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} - \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} + \frac{m^2}{a_3^2}, \quad (2.6d)$$

$$G_3^0 = m \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} \right). \quad (2.6e)$$

### B. Spinor field

For a spinor field  $\psi$ , the symmetry between  $\psi$  and  $\bar{\psi}$  appears to demand that one should choose the symmetrized Lagrangian [60]. Keeping this in mind we choose the spinor field Lagrangian as [24]:

$$L_{\text{sp}} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - m_{\text{sp}} \bar{\psi} \psi - F, \quad (2.7)$$

where the nonlinear term  $F$  describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field. Here we consider  $F = F(K)$  with  $K$  being one of the following expressions:  $\{I, J, I+J, I-J\}$ , where  $I = S^2 = (\bar{\psi} \psi)^2$  and  $J = P^2 = (\iota \bar{\psi} \gamma^5 \psi)^2$ . It can be shown that thanks to Fierz identity this type of nonlinear term describes the nonlinearity in its most general form. In (2.7)  $\nabla_\mu$  is the covariant derivative of spinor field:

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^\mu} + \bar{\psi} \Gamma_\mu, \quad (2.8)$$

with  $\Gamma_\mu$  being the spinor affine connection. In (2.7)  $\gamma$ 's are the Dirac matrices in curve spacetime and obey the following algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (2.9)$$

and are connected with the flat spacetime Dirac matrices  $\bar{\gamma}$  in the following way

$$g_{\mu\nu}(x) = e_\mu^a(x)e_\nu^b(x)\eta_{ab}, \quad \gamma_\mu(x) = e_\mu^a(x)\bar{\gamma}_a \quad (2.10)$$

where  $e_\mu^a$  is a set of tetrad 4-vectors.

For the metric (2.4) we choose the tetrad as follows:

$$e_0^{(0)} = 1, \quad e_1^{(1)} = a_1 e^{-mx_3}, \quad e_2^{(2)} = a_2, \quad e_3^{(3)} = a_3. \quad (2.11)$$

The Dirac matrices  $\gamma^\mu(x)$  of Bianchi type-III spacetime are connected with those of Minkowski one as follows:

$$\begin{aligned} \gamma^0 &= \bar{\gamma}^0, \quad \gamma^1 = \frac{e^{mx_3}}{a_1} \bar{\gamma}^1, \quad \gamma^2 = \frac{1}{a_2} \bar{\gamma}^2, \quad \gamma^3 = \frac{1}{a_3} \bar{\gamma}^3 \\ \gamma^5 &= -\iota \sqrt{-g} \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\iota \bar{\gamma}^0 \bar{\gamma}^1 \bar{\gamma}^2 \bar{\gamma}^3 = \bar{\gamma}^5 \end{aligned}$$

with

$$\bar{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \bar{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \bar{\gamma}^5 = \bar{\gamma}^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix},$$

where  $\sigma_i$  are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the  $\bar{\gamma}$  and the  $\sigma$  matrices obey the following properties:

$$\begin{aligned} \bar{\gamma}^i \bar{\gamma}^j + \bar{\gamma}^j \bar{\gamma}^i &= 2\eta^{ij}, \quad i, j = 0, 1, 2, 3 \\ \bar{\gamma}^i \bar{\gamma}^5 + \bar{\gamma}^5 \bar{\gamma}^i &= 0, \quad (\bar{\gamma}^5)^2 = I, \quad i = 0, 1, 2, 3 \\ \sigma^j \sigma^k &= \delta_{jk} + i\epsilon_{jkl} \sigma^l, \quad j, k, l = 1, 2, 3 \end{aligned}$$

where  $\eta_{ij} = \{1, -1, -1, -1\}$  is the diagonal matrix,  $\delta_{jk}$  is the Kronekar symbol and  $\epsilon_{jkl}$  is the totally antisymmetric matrix with  $\epsilon_{123} = +1$ .

The spinor affine connection matrices  $\Gamma_\mu(x)$  are uniquely determined up to an additive multiple of the unit matrix by the equation

$$\frac{\partial \gamma_\nu}{\partial x^\mu} - \Gamma_{\nu\mu}^\rho \gamma_\rho - \Gamma_\mu \gamma_\nu + \gamma_\nu \Gamma_\mu = 0, \quad (2.12)$$

with the solution

$$\Gamma_\mu = \frac{1}{4} \bar{\gamma}_a \gamma^\nu \partial_\mu e_\nu^{(a)} - \frac{1}{4} \gamma_\rho \gamma^\nu \Gamma_{\mu\nu}^\rho. \quad (2.13)$$

From the Bianchi type-VI metric (2.13) one finds the following expressions for spinor affine connections:

$$\Gamma_0 = 0, \quad (2.14a)$$

$$\Gamma_1 = \frac{1}{2} \left( \dot{a}_1 \bar{\gamma}^1 \bar{\gamma}^0 - m \frac{a_1}{a_3} \bar{\gamma}^1 \bar{\gamma}^3 \right) e^{-mx_3}, \quad (2.14b)$$

$$\Gamma_2 = \frac{1}{2} \dot{a}_2 \bar{\gamma}^2 \bar{\gamma}^0, \quad (2.14c)$$

$$\Gamma_3 = \frac{1}{2} \dot{a}_3 \bar{\gamma}^3 \bar{\gamma}^0. \quad (2.14d)$$

### C. Field equations

Variation of (2.1) with respect to the metric function  $g_{\mu\nu}$  gives the Einstein field equation

$$G_{\mu}^{\nu} = R_{\mu}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R = -\kappa T_{\mu}^{\nu}, \quad (2.15)$$

where  $R_{\mu}^{\nu}$  and  $R$  are the Ricci tensor and Ricci scalar, respectively. Here  $T_{\mu}^{\nu}$  is the energy momentum tensor of the spinor field.

Varying (2.7) with respect to  $\bar{\psi}(\psi)$  one finds the spinor field equations:

$$\iota\gamma^{\mu}\nabla_{\mu}\psi - m_{\text{sp}}\psi - \mathcal{D}\psi - \iota\mathcal{G}\gamma^5\psi = 0, \quad (2.16a)$$

$$\iota\nabla_{\mu}\bar{\psi}\gamma^{\mu} + m_{\text{sp}}\bar{\psi} + \mathcal{D}\bar{\psi} + \iota\mathcal{G}\bar{\psi}\gamma^5 = 0, \quad (2.16b)$$

where we denote  $\mathcal{D} = 2SF_K K_I$  and  $\mathcal{G} = 2PF_K K_J$ , with  $F_K = dF/dK$ ,  $K_I = dK/dI$  and  $K_J = dK/dJ$ . In view of (2.16) the spinor field Lagrangian (2.7) can be rewritten as

$$\begin{aligned} L_{\text{sp}} &= \frac{\iota}{2} [\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi] - m_{\text{sp}}\bar{\psi}\psi - F(I, J) \\ &= \frac{\iota}{2} \bar{\psi}[\gamma^{\mu}\nabla_{\mu}\psi - m_{\text{sp}}\psi] - \frac{\iota}{2} [\nabla_{\mu}\bar{\psi}\gamma^{\mu} + m_{\text{sp}}\bar{\psi}]\psi - F(I, J), \\ &= 2(IF_I + JF_J) - F = 2KF_K - F(K). \end{aligned} \quad (2.17)$$

### D. Energy momentum tensor of the spinor field

The energy-momentum tensor of the spinor field is given by

$$T_{\mu}^{\rho} = \frac{\iota}{4}g^{\rho\nu} \left( \bar{\psi}\gamma_{\mu}\nabla_{\nu}\psi + \bar{\psi}\gamma_{\nu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma_{\nu}\psi - \nabla_{\nu}\bar{\psi}\gamma_{\mu}\psi \right) - \delta_{\mu}^{\rho}L_{\text{sp}}. \quad (2.18)$$

Then in view of (2.8) and (2.17) the energy-momentum tensor of the spinor field can be written as

$$\begin{aligned} T_{\mu}^{\rho} &= \frac{\iota}{4}g^{\rho\nu} (\bar{\psi}\gamma_{\mu}\partial_{\nu}\psi + \bar{\psi}\gamma_{\nu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma_{\nu}\psi - \partial_{\nu}\bar{\psi}\gamma_{\mu}\psi) \\ &\quad - \frac{\iota}{4}g^{\rho\nu} \bar{\psi}(\gamma_{\mu}\Gamma_{\nu} + \Gamma_{\nu}\gamma_{\mu} + \gamma_{\nu}\Gamma_{\mu} + \Gamma_{\mu}\gamma_{\nu})\psi - \delta_{\mu}^{\rho}(2KF_K - F(K)). \end{aligned} \quad (2.19)$$

As is seen from (2.19), is case if for a given metric  $\Gamma_{\mu}$ 's are different, there arise nontrivial non-diagonal components of the energy momentum tensor.

We consider the case when the spinor field depends on  $t$  only. Then after a little manipulations

from (2.19) for the components of the energy momentum tensor one finds:

$$T_0^0 = m_{\text{sp}} S + F(K), \quad (2.20a)$$

$$T_1^1 = T_2^2 = T_3^3 = F(K) - 2KF_K, \quad (2.20b)$$

$$T_1^0 = 0, \quad (2.20c)$$

$$T_2^0 = -\frac{\iota}{4} m \frac{a_2}{a_3} \bar{\psi} \tilde{\gamma}^2 \tilde{\gamma}^3 \tilde{\gamma}^0 \psi = -\frac{1}{4} m \frac{a_2}{a_3} A^1, \quad (2.20d)$$

$$T_3^0 = 0, \quad (2.20e)$$

$$\begin{aligned} T_2^1 &= \frac{\iota}{4} \frac{a_2}{a_1} e^{mx_3} \left[ \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right) \bar{\psi} \tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^0 \psi - \frac{m}{a_3} \bar{\psi} \tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^3 \psi \right] \\ &= \frac{1}{4} \frac{a_2}{a_1} e^{mx_3} \left[ \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right) A^3 - \frac{m}{a_3} A^0 \right], \end{aligned} \quad (2.20f)$$

$$T_3^1 = \frac{\iota}{4} \frac{a_3}{a_1} e^{mx_3} \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} \right) \bar{\psi} \tilde{\gamma}^3 \tilde{\gamma}^1 \tilde{\gamma}^0 \psi = \frac{1}{4} \frac{a_3}{a_1} e^{mx_3} \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} \right) A^2 \quad (2.20g)$$

$$T_3^2 = \frac{\iota}{4} \frac{a_3}{a_2} \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} \right) \bar{\psi} \tilde{\gamma}^2 \tilde{\gamma}^3 \tilde{\gamma}^0 \psi = \frac{1}{4} \frac{a_3}{a_2} \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} \right) A^1. \quad (2.20h)$$

It can be shown that bilinear spinor forms the obey the following system of equations:

$$\dot{S}_0 + \mathcal{G} A_0^0 = 0, \quad (2.21a)$$

$$\dot{P}_0 - \Phi A_0^0 = 0, \quad (2.21b)$$

$$\dot{A}_0^0 - \frac{m}{a_3} A_0^3 + \Phi P_0 - \mathcal{G} S_0 = 0, \quad (2.21c)$$

$$\dot{A}_0^3 - \frac{m}{a_3} A_0^0 = 0, \quad (2.21d)$$

$$\dot{v}_0^0 - \frac{m}{a_3} v_0^3 = 0, \quad (2.21e)$$

$$\dot{v}_0^3 - \frac{m}{a_3} v_0^0 + \Phi Q_0^{30} + \mathcal{G} Q_0^{21} = 0, \quad (2.21f)$$

$$\dot{Q}_0^{30} - \Phi v_0^3 = 0, \quad (2.21g)$$

$$\dot{Q}_0^{21} - \mathcal{G} v_0^3 = 0, \quad (2.21h)$$

where we denote where we denote  $S_0 = SV$ ,  $P_0 = PV$ ,  $A_0^\mu = A_\mu V$ ,  $v_0^\mu = v^\mu V$ ,  $Q_0^{\mu\nu} = Q^{\mu\nu} V$  and  $\Phi = m_{\text{sp}} + \mathcal{D}$ . Here  $S = \bar{\psi}\psi$  is a scalar,  $P = \iota \bar{\psi} \gamma^5 \psi$  is a pseudoscalar,  $v^\mu = \bar{\psi} \gamma^\mu \psi$  - vector,  $A^\mu = \bar{\psi} \gamma^5 \gamma^\mu \psi$  - pseudovector, and  $Q^{\mu\nu} = \bar{\psi} \sigma^{\mu\nu} \psi$  is antisymmetric tensor. In (2.21)  $V$  is the volume scale which is defined as

$$V = a_1 a_2 a_3. \quad (2.22)$$

Combining these equations together and taking the first integral one gets

$$(S_0)^2 + (P_0)^2 + (A_0^0)^2 - (A_0^3)^2 = C_1 = \text{Const}, \quad (2.23a)$$

$$(Q_0^{30})^2 + (Q_0^{21})^2 + (v_0^3)^2 - (v_0^0)^2 = C_2 = \text{Const} \quad (2.23b)$$

Now let us consider the Einstein field equations. In view of (2.6) and (2.20) with find the following system of Einstein Equations

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} = \kappa(F(K) - 2KF_K), \quad (2.24a)$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} - \frac{m^2}{a_3^2} = \kappa(F(K) - 2KF_K), \quad (2.24b)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} = \kappa(F(K) - 2KF_K), \quad (2.24c)$$

$$\frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} + \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} - \frac{m^2}{a_3^2} = \kappa(m_{\text{sp}}S + F(K)), \quad (2.24d)$$

$$m\left(\frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1}\right) = 0, \quad (2.24e)$$

$$0 = \frac{1}{4}m\frac{a_2}{a_3}A^1, \quad (2.24f)$$

$$0 = \frac{1}{4}\frac{a_2}{a_1}e^{mx_3}\left[\left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2}\right)A^3 - \frac{m}{a_3}A^0\right], \quad (2.24g)$$

$$0 = \frac{1}{4}\frac{a_3}{a_1}e^{mx_3}\left(\frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1}\right)A^2, \quad (2.24h)$$

$$0 = \frac{1}{4}\frac{a_3}{a_2}\left(\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3}\right)A^1. \quad (2.24i)$$

From (2.24f) one immediately finds

$$A^1 = 0, \quad (2.25)$$

whereas from (2.24e) one finds the following relation between  $a_1$  and  $a_3$ :

$$a_3 = X_0 a_1, \quad X_0 = \text{const.} \quad (2.26)$$

In view of (2.25) the relations (2.24i) fulfill even without imposing restrictions on the metric functions, whereas (2.24h) fulfills thanks to (2.24e). From (2.24g) one finds the following relations between  $A^0$  and  $A^3$ :

$$\left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2}\right)A^3 = \frac{m}{a_3}A^0. \quad (2.27)$$

Inserting (2.27) into (2.21d) one finds

$$\frac{\dot{A}_0^3}{A_0^3} = \left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2}\right), \quad (2.28)$$

with the solution

$$A_0^3 = X_{03}\left(\frac{a_1}{a_2}\right), \quad X_{03} = \text{const.} \quad (2.29)$$

As it was found in previous papers, due to explicit presence of  $a_3$  in the Einstein equations, one needs some additional conditions. In an early work we propose two different situation, namely, set  $a_3 = \sqrt{V}$  and  $a_3 = V$  which allows us to obtain exact solutions for the metric functions.

In a recent paper we imposed the proportionality condition, widely used in literature. Demanding that the expansion is proportion to a component of the shear tensor, namely

$$\vartheta = N_3 \sigma_3^3, \quad N_3 = \text{const.} \quad (2.30)$$

The motivation behind assuming this condition is explained with reference to Thorne [61], the observations of the velocity-red-shift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic today within  $\approx 30$  per cent [62, 63]. To put more precisely, red-shift studies place the limit

$$\frac{\sigma}{H} \leq 0.3, \quad (2.31)$$

on the ratio of shear  $\sigma$  to Hubble constant  $H$  in the neighborhood of our Galaxy today. Collins et al. [64] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfies that the condition  $\frac{\sigma}{\theta}$  is constant. Under this proportionality condition it was also found that the energy-momentum distribution of the model is strictly isotropic, which is absolutely true for our case.

In order to exploit the proportionality condition (2.30) Let us now find expansion and shear for BIII metric. The expansion is given by

$$\vartheta = u^\mu_{;\mu} = u^\mu_\mu + \Gamma^\mu_{\mu\alpha} u^\alpha, \quad (2.32)$$

and the shear is given by

$$\sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu}, \quad (2.33)$$

with

$$\sigma_{\mu\nu} = \frac{1}{2} [u_{\mu;\alpha} P^\alpha_\nu + u_{\nu;\alpha} P^\alpha_\mu] - \frac{1}{3} \vartheta P_{\mu\nu}, \quad (2.34)$$

where the projection vector  $P$ :

$$P^2 = P, \quad P_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu, \quad P^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu. \quad (2.35)$$

In comoving system we have  $u^\mu = (1, 0, 0, 0)$ . In this case one finds

$$\vartheta = \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} = \frac{\dot{V}}{V}, \quad (2.36)$$

and

$$\sigma_1^1 = -\frac{1}{3} \left( -2 \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right) = \frac{\dot{a}_1}{a_1} - \frac{1}{3} \vartheta, \quad (2.37a)$$

$$\sigma_2^2 = -\frac{1}{3} \left( -2 \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} + \frac{\dot{a}_1}{a_1} \right) = \frac{\dot{a}_2}{a_2} - \frac{1}{3} \vartheta, \quad (2.37b)$$

$$\sigma_3^3 = -\frac{1}{3} \left( -2 \frac{\dot{a}_3}{a_3} + \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} \right) = \frac{\dot{a}_3}{a_3} - \frac{1}{3} \vartheta. \quad (2.37c)$$

One then finds

$$\sigma^2 = \frac{1}{2} \left[ \sum_{i=1}^3 \left( \frac{\dot{a}_i}{a_i} \right)^2 - \frac{1}{3} \vartheta^2 \right] = \frac{1}{2} \left[ \sum_{i=1}^3 H_i^2 - \frac{1}{3} \vartheta^2 \right]. \quad (2.38)$$

Inserting (2.26) into (2.36), (2.37) and (2.38) we find

$$\vartheta = 2 \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2}, \quad (2.39)$$

and

$$\sigma_1^1 = \frac{1}{3} \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right), \quad (2.40a)$$

$$\sigma_2^2 = -\frac{2}{3} \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right), \quad (2.40b)$$

$$\sigma_3^3 = \frac{1}{3} \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right). \quad (2.40c)$$



On account of (2.26), (2.37c), (2.22) from (2.30) one finds

$$a_1 = \left[ \frac{1}{X_0 X_1} V \right]^{\frac{1}{3} + \frac{1}{N_3}}, \quad a_2 = X_1 \left[ \frac{1}{X_0 X_1} V \right]^{\frac{1}{3} - \frac{2}{N_3}}, \quad a_3 = X_0 \left[ \frac{1}{X_0 X_1} V \right]^{\frac{1}{3} + \frac{1}{N_3}}, \quad (2.41)$$

where  $X_1$  is an integration constant. As one sees from (2.41), the isotropization process can take place only for  $N_3 \gg 3$ .

The equation for  $V$  can be found from the Einstein Equation (2.6) which for some manipulation looks

$$\ddot{V} = \bar{X} V^{\frac{1}{3} - \frac{2}{N_3}} + \frac{3\kappa}{2} [m_{\text{sp}} S + 2(F(K) - K F_K)] V, \quad \bar{X} = 2m^2 X_0^{\frac{2}{N_3} - \frac{4}{3}} X_1^{\frac{2}{N_3} + \frac{2}{3}}. \quad (2.42)$$

In order to solve (2.42) we have to know the relation between the spinor and the gravitational fields. Let us first find those relations for different  $K$ .

In case of  $K = I$ , i.e.  $\mathcal{G} = 0$  from (2.21a) we duly have

$$\dot{S}_0 = 0, \quad (2.43)$$

with the solution

$$S = \frac{V_0}{V}, \quad \Rightarrow \quad K = I = S^2 = \frac{V_0^2}{V^2}, \quad V_0 = \text{const}. \quad (2.44)$$

In this case spinor field can be either massive or massless.

As far as case with  $K = \{J, I + J, I - J\}$  that gives  $K_J = \pm 1$  is concerned, it can be solved exactly only for a massless spinor field.

In case of  $K = J$ , i.e.  $\Phi = \mathcal{D} = 0$  from (2.21b) we duly have

$$\dot{P}_0 = 0, \quad (2.45)$$

with the solution

$$P = \frac{V_0}{V}, \quad \Rightarrow \quad K = J = P^2 = \frac{V_0^2}{V^2}, \quad V_0 = \text{const}. \quad (2.46)$$

In case of  $K = I + J$  the equations (2.21a) and (2.21b) can be rewritten as

$$\dot{S}_0 + 2P F_K A_0^0 = 0, \quad (2.47a)$$

$$\dot{P}_0 - 2S F_K A_0^0 = 0, \quad (2.47b)$$

which can be rearranged as

$$S_0 \dot{S}_0 + P_0 \dot{P}_0 = \frac{d}{dt} (S_0^2 + P_0^2) = \frac{d}{dt} (V^2 K) = 0, \quad (2.48)$$

with the solution

$$K = I + J = \frac{V_0^2}{V^2}, \quad V_0 = \text{const}. \quad (2.49)$$

Note that one can represent  $S$  and  $P$  as follows:

$$S = \frac{V_0}{V} \sin \theta, \quad P = \frac{V_0}{V} \cos \theta. \quad (2.50)$$

The term  $\theta$  can be determined from (2.47a) or (2.47b) on account of (2.27), (2.29) and (2.41). It can be shown that  $\theta = \theta(V)$ .

Finally, for  $K = I - J$  the equations (2.21a) and (2.21b) can be rewritten as

$$\dot{S}_0 - 2PF_K A_0^0 = 0, \quad (2.51a)$$

$$\dot{P}_0 - 2SF_K A_0^0 = 0, \quad (2.51b)$$

which can be rearranged as

$$S_0 \dot{S}_0 - P_0 \dot{P}_0 = \frac{d}{dt}(S_0^2 - P_0^2) = \frac{d}{dt}(V^2 K) = 0, \quad (2.52)$$

with the solution

$$K = I - J = \frac{V_0^2}{V^2}, \quad V_0 = \text{const.} \quad (2.53)$$

As in previous case one can rewrite  $S$  and  $P$  as follows:

$$S = \frac{V_0}{V} \cosh \theta, \quad P = \frac{V_0}{V} \sinh \theta. \quad (2.54)$$

As one sees the non-triviality of non-diagonal components of the energy-momentum tensors, namely  $T_2^1$ ,  $T_3^2$  and  $T_3^1$  is directly connected with the affine spinor connections  $\Gamma_i$ 's.

### III. SOLUTION TO THE FIELD EQUATIONS

In this section we solve the field equations. Let us begin with the spinor field equations. In view of (2.8) and (2.14) the spinor field equation (2.16a) takes the form

$$\iota \tilde{\gamma}^0 \left( \dot{\psi} + \frac{1}{2} \frac{\dot{V}}{V} \psi \right) - m_{\text{sp}} \psi - \frac{m}{2a_3} \tilde{\gamma}^3 \psi - \mathcal{D} \psi - \iota \mathcal{G} \tilde{\gamma}^5 \psi = 0, \quad (3.1a)$$

$$\iota \left( \dot{\bar{\psi}} + \frac{1}{2} \frac{\dot{V}}{V} \bar{\psi} \right) \tilde{\gamma}^0 + m_{\text{sp}} \bar{\psi} - \frac{m}{2a_3} \bar{\psi} \tilde{\gamma}^3 + \mathcal{D} \bar{\psi} + \iota \mathcal{G} \bar{\psi} \tilde{\gamma}^5 = 0. \quad (3.1b)$$

As we have already mentioned,  $\psi$  is a function of  $t$  only. We consider the 4-component spinor field given by

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \quad (3.2)$$

Denoting  $\phi_i = \sqrt{V} \psi_i$  and  $\bar{X}_0 = m X_0^{1/N_3 - 2/3} X_1^{1/N_3 + 1/3}$  from (3.1) for the spinor field we find we

$$\dot{\phi}_1 + \iota \Phi \phi_1 + \left[ \iota \frac{\bar{X}_0}{2V^{1/3+1/N_3}} + \mathcal{G} \right] \phi_3 = 0, \quad (3.3a)$$

$$\dot{\phi}_2 + \iota \Phi \phi_2 - \left[ \iota \frac{\bar{X}_0}{2V^{1/3+1/N_3}} - \mathcal{G} \right] \phi_4 = 0, \quad (3.3b)$$

$$\dot{\phi}_3 - \iota \Phi \phi_3 + \left[ \iota \frac{\bar{X}_0}{2V^{1/3+1/N_3}} - \mathcal{G} \right] \phi_1 = 0, \quad (3.3c)$$

$$\dot{\phi}_4 - \iota \Phi \phi_4 - \left[ \iota \frac{\bar{X}_0}{2V^{1/3+1/N_3}} + \mathcal{G} \right] \phi_2 = 0. \quad (3.3d)$$

Further denoting  $\mathcal{Y} = \frac{\bar{X}_0}{2V^{1/3+1/N_3}}$  we can write the foregoing system of equation in the form:

$$\dot{\phi} = A\phi, \quad (3.4)$$

with  $\phi = \text{col}(\phi_1, \phi_2, \phi_3, \phi_4)$  and

$$A = \begin{pmatrix} -\iota\Phi & 0 & -\iota\mathcal{Y} - \mathcal{G} & 0 \\ 0 & -\iota\Phi & 0 & \iota\mathcal{Y} - \mathcal{G} \\ -\iota\mathcal{Y} + \mathcal{G} & 0 & \iota\Phi & 0 \\ 0 & \iota\mathcal{Y} + \mathcal{G} & 0 & \iota\Phi \end{pmatrix}. \quad (3.5)$$

It can be easily found that

$$\det A = (\Phi^2 + \mathcal{Y}^2 + \mathcal{G}^2)^2. \quad (3.6)$$

The solution to the equation (3.4) can be written in the form The solution to the equation (3.4) can be written in the form

$$\phi(t) = \text{Texp}\left(-\int_t^{t_1} A_1(\tau)d\tau\right)\phi(t_1), \quad (3.7)$$

where

$$A_1 = \begin{pmatrix} -\iota\mathcal{D} & 0 & -\iota\mathcal{Y} - \mathcal{G} & 0 \\ 0 & -\iota\mathcal{D} & 0 & \iota\mathcal{Y} - \mathcal{G} \\ -\iota\mathcal{Y} + \mathcal{G} & 0 & \iota\mathcal{D} & 0 \\ 0 & \iota\mathcal{Y} + \mathcal{G} & 0 & \iota\mathcal{D} \end{pmatrix}. \quad (3.8)$$

and  $\phi(t_1)$  is the solution at  $t = t_1$ . As we have already shown,  $K = V_0^2/V^2$  for  $K = \{J, I+J, I-J\}$  with trivial spinor-mass and  $K = V_0^2/V^2$  for  $K = I$  for any spinor-mass. Since our Universe is expanding, the quantities  $\mathcal{D}$ ,  $\mathcal{Y}$  and  $\mathcal{G}$  become trivial at large  $t$ . Hence in case of  $K = I$  with non-trivial spinor-mass one can assume  $\phi(t_1) = \text{col}(e^{-im_{\text{sp}}t_1}, e^{-im_{\text{sp}}t_1}, e^{im_{\text{sp}}t_1}, e^{im_{\text{sp}}t_1})$ , whereas for other cases with trivial spinor-mass we have  $\phi(t_1) = \text{col}(\phi_1^0, \phi_2^0, \phi_3^0, \phi_4^0)$  with  $\phi_i^0$  being some constants. Here we have used the fact that  $\Phi = m_{\text{sp}} + \mathcal{D}$ . The other way to solve the system (3.3) is given in [27].

As far as equation for  $V$ , i.e., (2.42) is concerned, we solve it setting  $K = I$  as in this case we can use the mass term as well. Assuming

$$F = \sum_k \lambda_k I^{n_k} = \sum_k \lambda_k S^{2n_k} \quad (3.9)$$

on account of  $S = V_0/V$  we find

$$\ddot{V} = \Phi_1(V), \quad \Phi_1(V) = \bar{X}V^{\frac{1}{3}-\frac{2}{N_3}} + \frac{3\kappa}{2} [m_{\text{sp}}V_0 + 2 \sum_k \lambda_k (1 - n_k) V_0^{2n_k} V^{1-2n_k}]. \quad (3.10)$$

To determine the type of nonlinearity that can be dominant both at the early stage as well as late time of evolution let us go back to (3.10). As one sees, for the nonlinearity to be dominant at early stage when  $V \rightarrow 0$  one should have  $n_k = n_1$ :  $n_1 > 1/2$  and  $n_1 > 1/3 + 1/N_3$ . For  $n_k = n_0$ :  $n_0 = 1/2$  this term can be added to the mass term. And finally for the nonlinear term to prevail at late time when  $V \rightarrow \infty$  one should choose  $n_k = n_2$ :  $n_2 < 1/2$  and  $n_2 < 1/3 + 1/N_3$ . Then we can rewrite the equation for  $V$  with the nonlinear term that determines both the early stage and the later stage of equation as follows

$$\ddot{V} = \Phi_1(V), \quad (3.11)$$

$$\Phi_1(V) = \bar{X}V^{\frac{1}{3}-\frac{2}{N_3}} + \frac{3\kappa}{2} [(m_{\text{sp}} + \lambda_0)V_0 + 2\lambda_1(1 - n_1)V_0^{2n_1}V^{1-2n_1} + 2\lambda_2(1 - n_2)V_0^{2n_2}V^{1-2n_2}].$$

with the first integral

$$\dot{V} = \Phi_2(V), \quad (3.12)$$

$$\Phi_2(V) = \sqrt{\bar{X}_1 V^{(4N_3-6)/3N_3} + 3\kappa \left[ (m_{\text{sp}} + \lambda_0)V_0 V + \lambda_1 V_0^{2n_1} V^{2(1-n_1)} + \lambda_2 V_0^{2n_2} V^{2(1-n_2)} \right] + \bar{C}},$$

where we denote  $\bar{X}_1 = 3N_3\bar{X}/(2N_3 - 3)$  and  $\bar{C}$  is the constant of integration. The solution for  $V$  can be written in quadrature as

$$\int \frac{dV}{\Phi_2(V)} = t + t_0, \quad t_0 = \text{const.} \quad (3.13)$$

In what follows we solve the Eqn. (3.10) numerically. In doing so we determine  $\dot{V}(0)$  from (3.12) for the given value of  $V(0)$ . To define whether the model allows decelerated or accelerated mode of expansion we also study the behavior of deceleration parameter  $q$  defines as

$$q = -\frac{V\ddot{V}}{\dot{V}^2} = -\frac{V\Phi(V)}{\Phi_1^2(V)}, \quad (3.14)$$

which in view of (3.11) and (3.12) reads

$$q = -\frac{\bar{X}V^{(4N_3-6)/3N_3} + \frac{3\kappa}{2} \left[ (m_{\text{sp}} + \lambda_0)V_0 V + 2\lambda_1(1-n_1)V_0^{2n_1}V^{2(1-n_1)} + 2\lambda_2(1-n_2)V_0^{2n_2}V^{2(1-n_2)} \right]}{\bar{X}_1 V^{(4N_3-6)/3N_3} + 3\kappa \left[ (m_{\text{sp}} + \lambda_0)V_0 V + \lambda_1 V_0^{2n_1} V^{2(1-n_1)} + \lambda_2 V_0^{2n_2} V^{2(1-n_2)} \right] + \bar{C}}. \quad (3.15)$$

From (3.15) it can be easily established that

$$\lim_{V \rightarrow \infty} q \rightarrow -(1-n_2) < 0, \quad \text{since } n_2 < 1/2. \quad (3.16)$$

Thus we see that spinor field nonlinearity generates late time acceleration of the Universe.

Finally let us see, what happens to EoS (energy of state) parameter. In view of (2.20a), (2.20b) and (3.9) for the EoS parameter  $\omega$  we find

$$\omega = \frac{p}{\varepsilon} = -\frac{T_1^1}{T_0^0} = \frac{\sum_k \lambda_k (2n_k - 1) S^{2n_k}}{\sum_k \lambda_k S^{2n_k} + m_{\text{sp}} S}, \quad (3.17)$$

which on account of discussions above can be rewritten as

$$\omega = \frac{\lambda_1 (2n_1 - 1) V_0^{2n_1} V^{2n_2} + \lambda_2 (2n_2 - 1) V_0^{2n_2} V^{2n_1}}{(\lambda_0 + m_{\text{sp}}) V_0 V^{2(n_1+n_2)-1} + \lambda_1 V_0^{2n_1} V^{2n_2} + \lambda_2 V_0^{2n_2} V^{2n_1}}. \quad (3.18)$$

Since we are interested in qualitative picture here, so we set the value of problem parameters very simple. Here we set  $m = 1$ ,  $X_1 = 1$ ,  $X_0 = 1$ ,  $V_0 = 1$ ,  $\lambda_0 = 1$ ,  $m_{\text{sp}} = 1$ ,  $C_0 = 1$ ,  $\kappa = 1$ ,  $N_3 = 3$ . We consider two cases for different combination with  $\lambda_1 = \pm 1$  and  $\lambda_2 = \pm 1$ . It was found that depending on the sign of  $\lambda_2$  the model provides two different type of solution, namely a positive  $\lambda_2$  gives rise to an expanding mode of evolution, whereas a negative  $\lambda_2$  generates oscillatory mode of evolution. In Figs. 1 and 2 we plotted the evolution of the Universe for a positive and negative value of  $\lambda_2$ , respectively. The sign of  $\lambda_1$  does not give a qualitatively different picture. In Fig. 3 we have plotted the dynamics of deceleration parameter  $q$  that shows a late time acceleration. In Fig. 4 we illustrated the EoS parameter for a positive  $\lambda_2$  that gives rise to an accelerated mode of expansion. As one sees it is positive at the beginning and becomes negative in the course of evolution which is in correspondence with present day observations. It should be noted that both deceleration and EoS parameters are time varying. This fact is also in agreement with the modern picture of the evolution of the Universe.

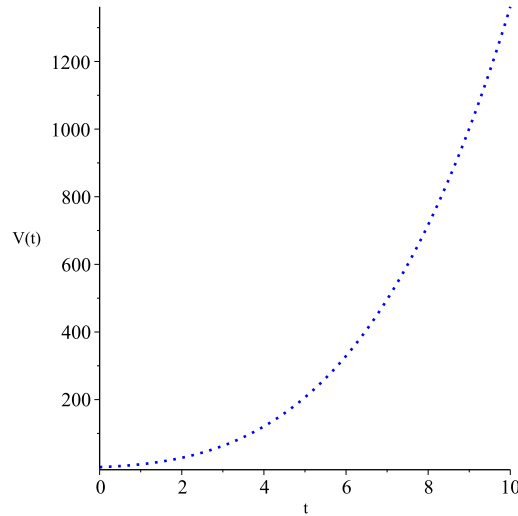


FIG. 1. Evolution of the Universe filled with massive spinor field for a positive  $\lambda_2$ . Here we set  $\lambda_1 = 1$  and  $\lambda_2 = 1$

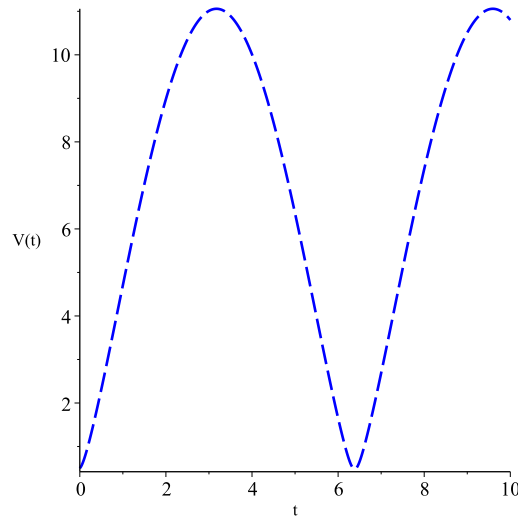


FIG. 2. Evolution of the Universe filled with massive spinor field for a negative  $\lambda_2$ . In this case we set  $\lambda_1 = -1$  and  $\lambda_2 = -1$

#### IV. CONCLUSION

Within the scope of Bianchi type-III spacetime we study the role of spinor field on the evolution of the Universe. Unlike Bianchi type I, V or  $VI_0$  cases where either both spinor mass and spinor field nonlinearity vanish [47–49] or the metric functions are similar to each other, i.e.,  $a_1 \sim a_2 \sim a_3$  in this case no such problem occurs. As one can see from (2.41) the spacetime remains locally rotationally symmetric and anisotropic all the time, though the isotropy of the spacetime can be achieved for a large proportionality constant. As far as evolution is concerned, depending on the sign of coupling constant the models allows both accelerated and oscillatory mode of expansion. A negative coupling constant leads to an oscillatory mode of expansion whereas a positive coupling

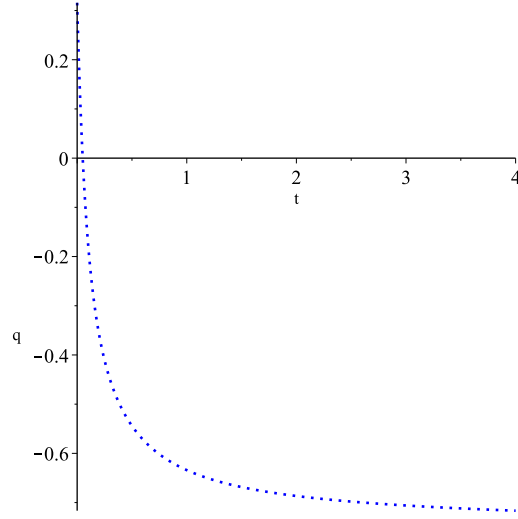


FIG. 3. Plot of deceleration parameter  $q$  for a positive  $\lambda_2$ . This case corresponds the evolution of volume scale given in Fig. 1

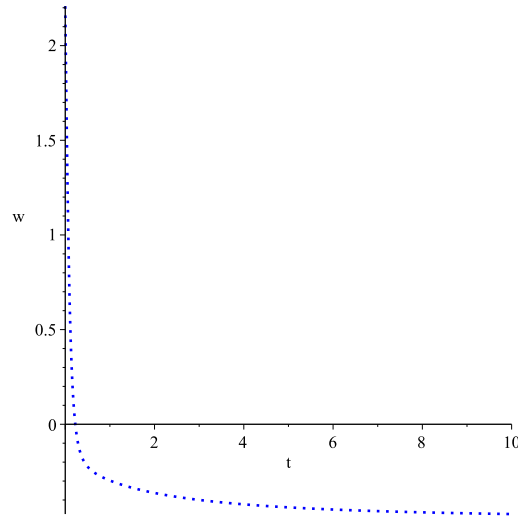


FIG. 4. Evolution of EoS parameter  $w$  for a positive  $\lambda_2$ . This case corresponds the evolution of volume scale given in Fig. 1

constants generates expanding Universe with late time acceleration. Both deceleration parameter and EoS parameter in this case vary with time and are in agreement with modern concept of spacetime evolution.

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